JOURNAL OF APPROXIMATION THEORY 76, 203–218 (1994)

Meromorphic and Entire Approximation in BMO-Norm*

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Communicated by Edward B. Saff

Received February 7, 1991; accepted in revised form August 17, 1992

A. Nersesjan and A. Roth proved the local character of the uniform meromorphic approximation on closed sets. In this paper the technique of A. Roth, based on the fusion lemma, is extended to BMO-norm. Thus, by reducing the approximation by meromorphic functions in BMO-norm to rational approximation in this norm on compact sets, the closed sets where the approximation by meromorphic functions is possible are characterized in terms of Hausdorff content. Sufficient conditions for approximation by entire functions are also obtained by using the technique of pushing poles in BMO-norm.

INTRODUCTION

Let F be a relatively closed subset of the complex plane $\mathbb{C}$. Denote respectively by $C(F)$, $H(F)$, $M_F(\mathbb{C})$ the set of continuous functions on F, the holomorphic functions on a neighbourhood of F, and the meromorphic functions in $\mathbb{C}$ without poles in F. If K is a compact subset of $\mathbb{C}$, then $R(K)$ denotes the set of rational functions with poles off K. A. Nersesjan and A. Roth proved the local character that has the uniform meromorphic approximation on closed sets. The technique of A. Roth, based on the fusion lemma [8], was extended to $\text{Lip}\,\alpha$ [4], and L^p [1] in order to study the meromorphic approximation on closed subsets in these norms.

In this paper we study the same problem but in BMO-norm. It is important to observe that our results are formally similar to the ones obtained for $\text{Lip}\,\alpha$ norms and L^p norms. In fact, $\text{BMO}(\mathbb{C})$ can be viewed as the natural limit of the $\text{Lip}\,\alpha$ spaces as $\alpha \to 0^+$; thus the BMO-fusion lemma, the localization theorem and the other results for approximation in BMO norms on closed subsets can also be interpreted as limiting statements of Fariña's results.

* Partially supported by Proyecto de Investigación DGICYT PS 89-0135.

0021-9045/94 $6.00

The paper is organized as follows. In Section 1, you will find the notation and definitions used throughout the paper with some preliminary results. In Section 2, we establish a version of Runge's theorem in BMO-norm, where it is shown that this theorem holds if we consider compact subsets of the Riemann Sphere ($\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$) and if the poles of rational approximating functions are fixed. In Section 3, the BMO-fusion lemma is proved. As a consequence of this fusion lemma we obtain in Section 4 a localization theorem in BMO that reduces the problem of approximating by meromorphic functions on a closed set to an approximation by rational functions on compact sets. Thus Verdera's theorem [10] and Runge's theorem in BMO are generalized to closed, possibly unbounded, subsets of $\mathbb{C}$. The generalization of Verdera's theorem involves the notion of one-dimensional Hausdorff content. Recently, by using constructive methods of localization of singularities, A. Boivin and J. Verdera [2] have proved a similar result with $M_F(\mathbb{C})$ replaced with $H(F)$. It is important to note that these results are actually equivalent. In fact, Runge's theorem on closed sets in BMO-norm states that any function of $H(F)$ may be approximated in BMO-norm by meromorphic functions having no poles on F. Finally, in Section 5, by using the pole shifting method based on the BMO-Runge's theorem we will obtain an extension of Arakelyan's theorem which gives sufficient conditions to the approximation in BMO-norm by entire functions.

1. Preliminaries

Let m be the Lebesgue measure on the complex plane $\mathbb{C}$. Let us fix $f \in L^1_{\mathrm{loc}}(\mathbb{C})$. The mean oscillation of f on a disc Δ is

$$M(f, \Delta) = \frac{1}{|\Delta|} \int |f(z) - f_\Delta| \, dm(z),$$

where $|\Delta| = m(\Delta)$ is the area of Δ and $f_\Delta = 1/|\Delta| \int_\Delta f(z)\, dm(z)$ is the mean value of f on Δ. We write $f \in \mathrm{BMO}(\mathbb{C})$ and we say that f is of bounded mean oscillation if

$$\|f\|_{0, \mathbb{C}} = \sup_\Delta M(f, \Delta) < \infty,$$

the supremum being taken over all discs Δ. $\|\cdot\|_0$ is a (complete) seminorm on $\mathrm{BMO}(\mathbb{C})$ vanishing only on constant functions.

For f in $\mathrm{BMO}(\mathbb{C})$ and $\delta > 0$, set

$$M_f(\delta) = \sup\{M(f, \Delta) \colon \text{radius } \Delta \leqslant \delta\}.$$

The space VMO($\mathbb{C}$) is the set of those functions $f \in \mathrm{BMO}(\mathbb{C})$ which have vanishing mean oscillation, that is , which satifsy $M_f(\delta) \to 0$ as $\delta \to 0^+$.

If F is a closed subset of $\mathbb{C}$, we define

$$\mathrm{BMO}(F) = \{f\colon F \to \mathbb{C}\colon \exists\, g \in \mathrm{BMO}(\mathbb{C}), f = g_{|F} \text{ a.e.}\}$$

and, for $f \in \mathrm{BMO}(F)$,

$$\|f\|_{0,F} = \inf\{\|g\|_{0,\mathbb{C}} : g \in \mathrm{BMO}(\mathbb{C}) \text{ and } g_{|F} = f \text{ a.e.}\}.$$

$\mathrm{BMO}_{\mathrm{loc}}(F)$, respectively $\mathrm{VMO}_{\mathrm{loc}}(F)$, consists of all the functions f on F which belong to $\mathrm{BMO}(E)$, respectively $\mathrm{VMO}(E)$, for all measurable bounded subsets $E \subset F$.

Let

$$A_0(F) = \mathrm{VMO}_{\mathrm{loc}}(F) \cap H(F^0).$$

Given a class of complex functions A, denote by $[A]_{0,F}$ the set of all the functions which are limits in BMO-norm on F of functions belonging to A. A result of P. J. Holden shows that $[H(F)]_{0,F} \subset A_0(F)$ [7].

A measure function is an increasing continuous function $h(t)$, $t \geqslant 0$, such that $h(0) = 0$. If h is a measure function and $E \subset \mathbb{C}$ we write

$$M^h(E) = \inf \sum_j h(\delta_j),$$

where the infimum is taken over all countable coverings of E by squares Q_j with sides of length δ_j, parallel to the coordinate axis. When $h(t) = t^\beta$, $\beta < 0$, $M^h(E) = M^\beta$ is called the β-dimensional Hausdorff content of E. The lower one-dimensional Hausdorff content of E is defined by

$$M^1_*(E) = \sup M^h(E),$$

the supremum being taken over all measure functions h which satisfy $h(t) \leqslant t$ and $h(t)\, t^{-1} \to 0$ as $t \to 0^+$. The set functions M^1 and M^1_* were used by Verdera to characterize the compact subsets of $\mathbb{C}$ for BMO rational approximation [10].

The letter C denotes, in this paper, a positive constant independent of the relevant variables under consideration and may be different at each occurrence.

We close this preliminary section by recalling a few known results which are used throughout the paper.

DEFINITION. A sequence of pairs $\{\Delta_j, \varphi_j\}$ is said to be an almost disjoint covering of an open subset Ω of the complex plane if there exists a constant C such that

(i) $\Omega = \bigcup_{j=1}^{\infty} \Delta_j$, Δ_j are discs with radii δ_j.

(ii) No point of Ω lies in more than C discs Δ_j.

(iii) $\{\varphi_j\}$ is a C^{∞}-partition of unity, subordinate to $\{\Delta_j\}$ with $\|\nabla\varphi_j\|_{\infty} < C/\delta_j$.

We say that a family $\{\Delta_j\}$ of discs of $\mathbb{C}$ is almost disjoint if (i) and (ii) are satisfied. Note that given an arbitrary open subset of $\mathbb{C}$, Ω then, according to [3] (see also [2, 6]), it is possible to find an almost disjoint covering of Ω. In particular, if we consider an almost disjoint family of discs of $\mathbb{C}$ one has the following lemma due to Boivin and Verdera [2, Lemma 1.7.3], the proof of which is given here for completeness.

LEMMA 1. *Let $\{h_j\}$ be a sequence of functions $h_j \in \mathrm{BMO}(\mathbb{C})$ such that the support of h_j is contained in Δ_j, where $\{\Delta_j\}$ is an almost disjoint family of discs of $\mathbb{C}$; then*

$$\left\|\sum_j h_j\right\|_{0,\,\mathbb{C}} \leqslant \max_j \|h_j\|_{0,\,\mathbb{C}}.$$

Proof. Let Δ be a disc of radius r and let δ_j be the radius of Δ_j. If we consider three subsets of $\mathbb{N}$

$$\mathrm{I} = \{j\colon \Delta_j \cap \Delta \neq \varnothing \text{ and } \Delta_j \subset 2\Delta\}$$

$$\mathrm{II} = \{j\colon \Delta_j \cap \Delta \neq \varnothing \text{ and } \Delta_j \backslash 2\Delta \neq \varnothing\}$$

$$\mathrm{III} = \{j\colon \Delta_j \cap \Delta = \varnothing\},$$

then we have

$$M\left(\sum_j h_j, \Delta\right) \leqslant \sum_{j\in \mathrm{I}} M(h_j, \Delta) + \sum_{j\in \mathrm{II}} M(h_j, \Delta).$$

Now if $j \in \mathrm{I}$ is fixed, by using the estimation (9) of [10, p. 295] with $N = r/\delta_j$, we obtain

$$M(h_j, \Delta) \leqslant C\left(\frac{r}{\delta_j}\right)^{-2} M(h_j, 2\Delta_j) \leqslant C\left(\frac{\delta_j}{r}\right)^2 \|h_j\|_{0,\,\mathbb{C}},$$

but since Δ_j is an almost disjoint covering,

$$\sum_{j\in \mathrm{I}} \delta_j^2 = \frac{1}{\pi} \sum_{j\in \mathrm{I}} |\Delta_j| \leqslant C\left|\bigcup_{j\in \mathrm{I}} \Delta_j\right| \leqslant C\,|\Delta| = Cr^2$$

and then

$$\sum_{j\in \mathrm{I}} M(h_j, \Delta) \leqslant C \max_{j\in \mathrm{I}} \|h_j\|_{0,\,\mathbb{C}} \leqslant C \max_j \|h_j\|_{0,\,\mathbb{C}}.$$

To obtain the estimation $\sum_{j\in \mathrm{II}} M(h_j, \Delta) \leqslant C \max_j \|h_j\|_{0,\mathbb{C}}$, it is enough to note that card(II) is bounded by an absolute constant. Indeed, let Γ be the boundary of $\frac{3}{2}\Delta$. Since $\{\Delta_j\}$ are almost disjoints and $\delta_j \geqslant r$, we have

$$3\pi r = \text{length}(\Gamma) > \text{length}\left(\bigcup_{j\in \mathrm{II}} \Delta_j \cap \Gamma\right) \geqslant C \sum_{j\in \mathrm{II}} \text{length}(\Delta_j \cap \Gamma)$$

$$\geqslant C \sum_{j\in \mathrm{II}} \delta_j \geqslant Cr\,\text{card}(\mathrm{II}).$$

Thus the proof of the lemma is finished.

Recall that to each $\phi \in C_0^1(\mathbb{C})$ one associates Vitushkin operator [6, p. 210]

$$T_\phi : L^1_{\text{loc}}(\mathbb{C}) \to L^1_{\text{loc}}(\mathbb{C})$$

defined by

$$T_\phi f(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\xi)-f(z)}{\xi - z}\,\bar{\partial}\phi(\xi)\,dm(\xi) = f(z)\,\phi(z) + \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\xi)}{\xi - z}\,\bar{\partial}\phi(\xi)\,dm(\xi).$$

Moreover if $\Delta = \Delta(z, \delta)$ and $\phi \in C_0^1(\Delta)$, one has

PROPOSITION 2 [10, Proposition 3.2]. *Let $f \in \mathrm{BMO}(\mathbb{C})$ and $\phi \in C_0^1(\Delta)$. Then*

$$\|T_\phi f\|_{0,\mathbb{C}} \leqslant C\delta\,\|\nabla\phi\|_\infty\,\|f\|_{0,3\Delta}.$$

The proof of this proposition contains two results that we recall in the next lemmas.

LEMMA 3. *If $f \in \mathrm{BMO}(\mathbb{C})$ and $\phi \in C_0^1(\Delta)$, then*

$$\|f\phi\|_{0,\mathbb{C}} \leqslant C\delta\,\|\nabla\phi\|_\infty\,\|f\|_{0,3\Delta}.$$

On the other hand, under the same conditions, the Cauchy transform of $f\bar{\partial}\phi$, $(f\bar{\partial}\phi)^\wedge$, is defined by

$$(f\bar{\partial}\phi)^\wedge\,(z) = \frac{1}{\pi}\int_{\mathbb{C}} \frac{f(\xi)}{\xi - z}\,\bar{\partial}\phi(\xi)\,dm(\xi);$$

thus we have

LEMMA 4. $\|(f\bar{\partial}\phi)^\wedge\|_{0,\mathbb{C}} \leqslant C\delta\,\|\nabla\phi\|_\infty\,\|f\|_{0,3\Delta}.$

2. Runge's Theorem in BMO-norms

The fact that if $f \in L^\infty(\mathbb{C})$ then $\|f\|_{0,\mathbb{C}} \leqslant 2\|f\|_{\infty,\mathbb{C}}$ implies clearly Runge's theorem in BMO-norm even if we work with compact subsets of $\mathbb{C}^*$:

THEOREM 5. *Let K be a compact subset in $\mathbb{C}^*$ and f be a holomorphic function in a neighborhood of K. Then given $\varepsilon > 0$, there exists a rational function $R(z)$ without poles in K such that*

$$\|f - R\|_{0,K} < \varepsilon.$$

Proof. By Runge's theorem in $\|\cdot\|_\infty$ for compact subsets of $\mathbb{C}^*$, there exists a rational function R such that $\|f - R\|_{\infty,K} < \varepsilon/2$. Denote by h an extension of $f - R$ to $L^\infty(\mathbb{C})$ such that $\|h\|_{\infty,\mathbb{C}} < \varepsilon/2$.

Since $h \in L^\infty(\mathbb{C})$,

$$\|h\|_{0,\mathbb{C}} \leqslant 2\|h\|_{\infty,\mathbb{C}}$$

and we obtain

$$\|f - R\|_{0,K} = \inf\{\|g\|_{0,\mathbb{C}} : g \in \mathrm{BMO}(\mathbb{C}) \text{ and } g_{|K} = f \text{ a.e.}\} \leqslant \|h\|_{0,\mathbb{C}} < \varepsilon.$$

Remark. If $\infty \in K$ and $K \neq \mathbb{C}^*$, then Runge's theorem follows from the case $\infty \notin K$ (see for example [5, p. 94]) by applying first a linear transformation.

Note that Runge's theorem in uniform norm allows a relocation of the poles of the approximating rational functions without affecting the approximation itself. It follows easily from the argument above that the same must be true in BMO-norms; in fact we have:

THEOREM 6 (BMO-RUNGE'S THEOREM). *Suppose K is a compact set in $\mathbb{C}^*$ and $\{\alpha_j\}$ is a set which contains one point in each component of $\mathbb{C}^* \setminus K$. If f is a holomorphic function in a neighborhood of K and $\varepsilon > 0$ there exists a rational function R, all of whose poles lie in the prescribed set $\{\alpha_j\}$, such that*

$$\|f - R\|_{0,K} < \varepsilon.$$

3. The Fusion Lemma in BMO-norms

This section also deals with approximation on compact sets. However, the BMO-fusion lemma which we prove will be the main tool to study the BMO-approximation on closed sets.

THEOREM 7 (BMO-FUSION LEMMA). *Suppose K_1, K_2, and K are compact sets in $\mathbb{C}^*$ such that $K_1 \cap K_2 = \emptyset$. Then there exists a constant C depending only on K_1 and K_2 with the following property: if r_1, r_2 are rational functions without poles in K such that*

$$\|r_1 - r_2\|_{0, K} < \varepsilon,$$

then there exists a third rational function r without poles in K such that

$$\|r - r_i\|_{0, K_i \cup K} \leqslant C\varepsilon, \qquad i = 1, 2. \tag{1}$$

Proof. If $K_1 \cap K = \emptyset$ or $K_2 \cap K = \emptyset$, then the theorem follows immediately from Runge's theorem in BMO. For this, suppose, for example, that $K_1 \cap K = \emptyset$. Define

$$f(z) = \begin{cases} r_1(z), & z \in K_1 \\ r_2(z), & z \in K_2 \cup K. \end{cases}$$

Then by BMO-Runge's theorem, there exists a rational function R such that

$$\|f - R\|_{0, K \cup K_1 \cup K_2} < \varepsilon.$$

Thus R satisfies (1). Then we may consider that $K_1 \cap K \neq \emptyset$ and $K_2 \cap K \neq \emptyset$. We also remark that it is enough to consider the case $r_2 \equiv 0$, since if $r_2 \not\equiv 0$, we let $\rho_1 = r_1 - r_2$ and $\rho_2 \equiv 0$. So if we assume that there exists a rational function ρ such that

$$\|\rho - \rho_i\|_{0, K \cup K_i} < \varepsilon, \qquad i = 1, 2,$$

then (1) is satisfied with $r = \rho + r_2$.

Without loss of generality, we can also assume that $\infty \in K_2$. We choose open neighborhoods U_1 and U_2 of K_1 and K_2, respectively, such that

(a) $U_1 \cap U_2 = \emptyset$

(b) $\partial U_1, \partial U_2 \in \mathscr{C}^1$.

Let $E = \mathbb{C} \setminus (U_1 \cup U_2)$ and note that E is a compact subset of $\mathbb{C}$. Since $r_1 \in R(K)$ and $\|r_1\|_{0, K} < \varepsilon$, then there exists a function $h \in \mathrm{BMO}(\mathbb{C})$ such that $h_{|K} = r_1$ and

$$\|h\|_{0, \mathbb{C}} < 2\, \|r_1\|_{0, K} < 2\varepsilon. \tag{2}$$

Now we define

$$f(z) = \begin{cases} h(z), & \text{if } z \in E \\ r_1(z), & \text{if } z \in \mathbb{C} \setminus E. \end{cases}$$

Let H be an infinitely differentiable function with compact support such that $H_{|U_1}=1$, $H_{|U_2}=0$ and $0 \leqslant H(z) \leqslant 1$, for all $z \in \mathbb{C}$, and set

$$F(z)=\frac{1}{\pi}\int_{\mathbb{C}}\frac{f(\xi)-f(z)}{\xi-z}\,\bar{\partial}H(\xi)\,dm(\xi)=f(z)\,H(z)+g(z),$$

where g is the Cauchy transform of $f\bar{\partial}H$ (i.e., $g=(f\bar{\partial}H)^{\wedge}$). It follows immediately from properties of the Vitushkin operator [6] that F is holomorphic in $K_1 \cup K_2 \cup K$, except for finitely many poles in K_1. Now, suppose that Δ is a disc with radius δ that contains the support of H and $G \in \mathrm{BMO}(\mathbb{C})$; then from Lemma 4

$$\|(G\bar{\partial}H)^{\wedge}\|_{0,\mathbb{C}}=C\delta\,\|\nabla H\|_{\infty}\,\|G\|_{0,3\Delta}.$$

Moreover if $G \in \mathrm{BMO}(\mathbb{C})$ and $G_{|E}=f$ a.e., then

$$\|g\|_{0,\mathbb{C}}=\|(f\bar{\partial}H)^{\wedge}\|_{0,\mathbb{C}}=\|(G\bar{\partial}H)^{\wedge}\|_{0,\mathbb{C}}\leqslant C\delta\,\|\nabla H\|_{\infty}\,\|G\|_{0,3\Delta}$$
$$\leqslant C\delta\,\|\nabla H\|_{\infty}\,\|G\|_{0,\mathbb{C}}$$

and since $\|f\|_{0,E}=\inf\{\|G\|_{0,\mathbb{C}}: G \in \mathrm{BMO}(\mathbb{C}) \text{ and } G_{|E}=f \text{ a.e.}\}$ one has

$$\|g\|_{0,\mathbb{C}}=\|(f\bar{\partial}H)^{\wedge}\|_{0,\mathbb{C}}\leqslant C\delta\,\|\nabla H\|_{\infty}\,\|f\|_{0,E}\leqslant C\,\|f\|_{0,E},$$

with C depending only on the choice of U_1 and U_2.

Thus, since $F-r_1=fH-r_1+g$ on $K_1 \cup K$, and $H \equiv 1$ on K_1, we have

$$\|F-r_1\|_{0,K_1\cup K}=\|fH+g-r_1\|_{0,K_1\cup K}\leqslant\|fH-r_1\|_{0,K_1\cup K}+\|g\|_{0,K_1\cup K}$$
$$=(\mathrm{I})+(\mathrm{II})$$

Let h be defined as in (2); then $(\mathrm{II})\leqslant C\,\|f\|_{0,E}=C\,\|h\|_{0,E}\leqslant C\,\|r_1\|_{0,K}$ and to estimate (I), note that $fH-r_1=(H-1)\,r_1=(H-1)\,h$ on $K \cup K_1$, that $(H-1)$ belongs to $\mathrm{BMO}(\mathbb{C})$, and that by Lemma 3

$$\|(H-1)\,h\|_{0,\mathbb{C}}\leqslant\|Hh\|_{0,\mathbb{C}}+\|h\|_{0,\mathbb{C}}\leqslant C\delta\,\|\nabla H\|_{\infty}\,\|r_1\|_{0,K}+2\,\|r_1\|_{0,K}$$
$$\leqslant C\,\|r_1\|_{0,K}.$$

Therefore, $(\mathrm{I})\leqslant C\,\|r_1\|_{0,K}$ and

$$\|F-r_1\|_{0,K_1\cup K}\leqslant C\,\|r_1\|_{0,K}.$$

Analogously, $F - r_2 = F = Hr_1 + g$ on K implies

$$\|F - r_2\|_{0, K_2 \cup K} = \|F\|_{0, K_2 \cup K} = \|fH + g\|_{0, K_2 \cup K} \leqslant \|fH\|_{0, K_2 \cup K} + \|g\|_{0, K_2 \cup K}$$
$$= (\mathrm{III}) + (\mathrm{IV}),$$

where

$$(\mathrm{IV}) \leqslant C\, \|f\|_{0, E} = C\, \|h\|_{0, E} \leqslant C\, \|r_1\|_{0, K}$$

and to estimate (III), by proceeding as above, we obtain, since $H \equiv 0$ on K_2,

$$\|Hh\|_{0, \mathbb{C}} \leqslant C\delta\, \|\nabla H\|_{\infty}\, \|r_1\|_{0, K} \leqslant C\, \|r_1\|_{0, K}.$$

Hence $(\mathrm{III}) \leqslant C\, \|r_1\|_{0, K}$ and

$$\|F - r_2\|_{0, K_2 \cup K} \leqslant C\, \|r_1\|_{0, K}.$$

Thus by applying the BMO-Runge's theorem to the function $F - \Sigma$ (where Σ is the sum of the principal parts of F in U_1), there exists a rational function r such that

$$\|F - r\|_{0, K \cup K_1 \cup K_2} < C\, \|r_1\|_{0, K}$$

and the theorem is proved.

4. Theorem of Localization

In this section, we show that the approximation in BMO-norm on a closed set F is equivalent to the approximation in BMO-norm on the compact subsets of F.

THEOREM 8. *Let f be a complex function on a closed subset F of $\mathbb{C}$. Then $f \in [M_F(\mathbb{C})]_{0, F}$, if and only if $f_{|F \cap \bar{G}_n} \in [R(F \cap \bar{G}_n)]_{0, F \cap \bar{G}_n}$, where $\{G_n\}$ is some exhausting sequence of $\mathbb{C}$ by bounded domains G_n such that $\bar{G}_n \subset G_{n+1}$ and $\bigcup G_n = \mathbb{C}$.*

Proof. Let $f \in [M_F(\mathbb{C})]_{0, F}$. Then $f_{|F \cap K} \in [R(F \cap K)]_{0, F \cap K}$ for each compact subset $K \subset \mathbb{C}$, since every function $g \in M_F(\mathbb{C})$ is analytic on $F \cap K$ and, by BMO-Runge's theorem, can be approximated on $F \cap K$ by rational functions in BMO-norms.

Now we consider a sequence of bounded domains $\{G_n\}_{n=1}^{\infty}$ which satisfies the hypotheses of theorem. For each $n=1,2,\ldots$ we apply the BMO-fusion lemma with

$$K_1=\bar{G}_n,\ K_2=\mathbb{C}^*\backslash G_{n+1} \qquad \text{and} \qquad F_n=F\cap\bar{G}_{n+1}.$$

We call the constant given by the fusion lemma A_n and without loss of generality assume $A_n\geqslant 1$ and increasing.

Let $\varepsilon>0$, by hypothesis $f_{|F_n}\in[R(F_n)]_{0,F_n}$; hence there exists a rational function q_n without poles on F_n such that

$$\|f-q_n\|_{0,F_n}<\frac{\varepsilon}{2^{n+1}A_n} \tag{3}$$

for every $n=1,2,\ldots$. Moreover, since $F_n\subset F_{n+1}$, we have

$$\|q_{n+1}-q_n\|_{0,F_n}<\frac{\varepsilon}{2^nA_n}. \tag{4}$$

By the BMO-fusion lemma there exists a rational function r_n such that

$$\|r_n-q_n\|_{0,F_n\cup K_1}<\frac{\varepsilon}{2^n} \tag{5}$$

and

$$\|r_n-q_{n+1}\|_{0,F_n\cup K_2}<\frac{\varepsilon}{2^n}. \tag{6}$$

We define

$$g(z)=q_1(z)+\sum_{n=1}^{\infty}(r_n(z)-q_n(z)), \tag{7}$$

and we claim that $g\in M_F(G)$, because $[H(\bar{G}_n)]_{0,\bar{G}_n}\subset A_0(\bar{G}_n)$ [7] and r_n-q_n can be chosen without poles on $\bar{G}_n$ (observe the construction of F and r in Theorem 7). Thus by (5), g has the same poles as $q_1(z)+\sum_{k<n}(r_n(z)-q_n(z))$ in G_n; hence there is no acumulation of poles in $\mathbb{C}$ and $g\in M_F(\mathbb{C})$.

Finally it only remains to prove that

$$\|f-g\|_{0,F}<\varepsilon.$$

To do this, it suffices to show

$$\|f-g\|_{0,F_n}<\varepsilon, \qquad n=1,2,\ldots \tag{8}$$

because if $g_n \in \mathrm{BMO}(\mathbb{C})$, $\|g_n\|_{0,\mathbb{C}} < \varepsilon$, and $g_{n|F_n} = g - f$, then we can consider an almost disjoint countable covering of the complex plane, $\{\Delta_j, \varphi_j\}$, Δ_j being of radii δ and define

$$m(z) = \sum_{j=1}^{\infty} g_j^* \varphi_j(z),$$

where g_j^* is equal to g_n for some n such that supp $\varphi_j \subset G_n$. Thus $m_{|F} = g - f$ a.e., and from Lemma 1

$$\|m\|_{0,\mathbb{C}} \leqslant \left\| \sum_{n=1}^{\infty} g_n^* \varphi_n \right\|_{0,\mathbb{C}} \leqslant C \max_n \|g_n^* \varphi_n\|_{0,\mathbb{C}}.$$

Moreover by Lemma 3

$$\|g_n^* \varphi_n\|_{0,\mathbb{C}} \leqslant C\delta \|\nabla \varphi_n\|_\infty \|g_n^*\|_{0,\mathbb{C}} \leqslant C \|g_n^*\|_{0,\mathbb{C}} < \varepsilon.$$

Therefore $m \in \mathrm{BMO}(\mathbb{C})$,

$$\|m\|_{0,\mathbb{C}} \leqslant C\varepsilon,$$

and

$$\|g - f\|_{0,F} \leqslant C\varepsilon.$$

To show (8), let $n = 1$; then by (3) and (5)

$$\|f - g\|_{0,F_1} \leqslant \|q_1 - f\|_{0,F_1} + \sum_{n=1}^{\infty} \|r_n - q_n\|_{0,F_1} < \varepsilon.$$

On the other hand, if $n > 1$ we have

$$\begin{aligned} \|f - g\|_{0,F_n} &\leqslant \sum_{k=1}^{n-1} \|r_k - q_{k+1}\|_{0,F_n} + \|q_n - f\|_{0,F_n} + \sum_{k=n}^{\infty} \|r_n - q_n\|_{0,F_n} \\ &\leqslant (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}). \end{aligned}$$

Since for $1 \leqslant k \leqslant n-1$, $F_n \subset (\mathbb{C}^* \setminus G_{k+1}) \cup F_k$,

$$(\mathrm{I}) = \sum_{k=1}^{n-1} \|r_k - q_{k+1}\|_{0,F_n} \leqslant \sum_{k=1}^{n-1} \frac{\varepsilon}{2^k},$$

by (3)

$$(\mathrm{II}) = \|q_n - f\|_{0,F_n} \leqslant \frac{\varepsilon}{2^{n+1}},$$

and for $k \geqslant n$, $F_n \subset F_k \cup \bar{G}_k$ and by (5)

$$(\mathrm{III}) \leqslant \sum_{k=n}^{\infty} \frac{\varepsilon}{2^k}.$$

Thus

$$\|f-g\|_{0, F_n} < \varepsilon,$$

and so the theorem is proved.

As a first application of this theorem we obtain the next corollary, which is a BMO-Runge's theorem for arbitrary closed subsets of $\mathbb{C}$.

COROLLARY 9. *Let F be a closed subset of $\mathbb{C}$ and $f \in H(F)$. Then f can be approximated in BMO-norm by functions in $M_F(\mathbb{C})$.*

Proof. Let $G_n = \Delta(0, n)$. Since F is a closed and $f \in H(F)$, we have $f \in H(F \cap \bar{G}_n)$ for all $n = 1, 2, \ldots$. Hence, $f \in [R(F \cap \bar{G}_n)]_{0, F \cap \bar{G}_n}$, and from Theorem 8 we infer that $f \in [M_F(\mathbb{C})]_{0, F}$.

The localization theorem also allows us to extend Verdera's theorem [10, Theorem 2] on the approximation of functions in $A_0(X)$ by functions in $R(X)$, when X is a compact subset of $\mathbb{C}$, to meromorphic approximation on arbitrary closed subsets of $\mathbb{C}$.

THEOREM 10. *Let F be a closed subset of $\mathbb{C}$. The following conditions are equivalent:*

(i) $A_0(F) = [M_F(\mathbb{C})]_{0, F}$

(ii) $M_*^1(\Delta(z, \delta) \backslash F^0) \leqslant C M^1(\Delta(z, \delta) \backslash F)$, *for all z and all open disc Δ.*

Proof. The proof of (i) $\Rightarrow$ (ii) is analogous to the compact case [10, Section 3]. To prove (ii) $\Rightarrow$ (i), let $\{G_n\}$ be an exhausing sequence of $\mathbb{C}$ by discs G_n such that $\bar{G}_n \subset G_{n+1}$ and $\bigcup G_n = \mathbb{C}$. According to Theorem 4, it is enough to show that

$$A_0(F \cap \bar{G}_n) = [R(F \cap \bar{G}_n)]_{0, F \cap \bar{G}_n}$$

and in virtue of [10, Theorem 2], this is equivalent to prove

$$M_*^1(\Delta(z, \delta) \backslash (F \cap \bar{G}_n)^0) \leqslant C M^1(\Delta(z, \delta) \backslash (F \cap \bar{G}_n)).$$

Denote $F \cap \bar{G}_n$ by X, and note that it follows from elementary properties of M_*^1 that

$$M_*^1(\Delta(z, \delta) \backslash X^0) \leqslant M_*^1(\Delta(z, \delta) \backslash X) + M_*^1(\Delta(z, \delta) \cap \partial X) = (\mathrm{I}).$$

Also recall that M_*^1 and M^1 agree on open sets and observe that $\partial X \cap \Delta(z, \delta)$ is contained in $(\partial F \cap \Delta(z, \delta)) \cup (\partial G_n)$, $\Delta(z, \delta) \backslash F \subset \Delta(z, \delta) \backslash X$ and that $M_*^1(\partial G_n) = 0$; so we obtain

$$\begin{aligned}(\mathrm{I}) &\leqslant M^1(\Delta(z, \delta) \backslash X) + M_*^1(\Delta(z, \delta) \cap \partial F) + M_*^1(\partial G_n)\\ &\leqslant M^1(\Delta(z, \delta) \backslash X) + M_*^1(\Delta(z, \delta) \backslash F^0) = (\mathrm{II}).\end{aligned}$$

Now by using (ii),

$$\begin{aligned}(\mathrm{II}) &\leqslant M^1(\Delta(z, \delta) \backslash X) + C M^1(\Delta(z, \delta) \backslash F)\\ &\leqslant M^1(\Delta(z, \delta) \backslash X) + C M^1(\Delta(z, \delta) \backslash X)\\ &\leqslant (C+1)\, M^1(\Delta(z, \delta) \backslash X);\end{aligned}$$

thus

$$M_*^1(\Delta(z, \delta) \backslash (F \cap \bar{G}_n)^0) \leqslant (C+1)\, M^1(\Delta(z, \delta) \backslash (F \cap \bar{G}_n)).$$

COROLLARY 11. *Let F be a closed nowhere dense subset of $\mathbb{C}$. Then $C(F) \subset [M_F(\mathbb{C})]_{0, F}$ if and only if there exists a constant $C > 0$ such that*

$$M^1(\Delta(z, \delta) \backslash F) \geqslant C\delta \tag{9}$$

for all z and all δ.

Proof. Let $f \in C(F)$. If $M^1(\Delta(z, \delta) \backslash F) \geqslant C\delta$ and $\{D_n\}$ is an exhaustion of $\mathbb{C}$ by open discs with $\bar{D}_n \subset D_{n+1}$, then $M^1(\Delta(z, \delta) \backslash F \cap \bar{D}_n) \geqslant C\delta$. Moreover by virtue of [10, Corollary of Theorem 2], $f \in R(F \cap \bar{D}_n)$ because $f \in C(F \cap \bar{D}_n)$. Hence by applying Theorem 10, $f \in [M_F(\mathbb{C})]_{0, F}$.

Conversely, let $f \in \mathrm{VMO}_{\mathrm{loc}}(F)$ and $\{D_n\}$ as above. Then $f \in \mathrm{VMO}(F \cap \bar{D}_n)$. Since $C(X)$ is dense in $\mathrm{VMO}(X)$ provided that X is a compact subset of $\mathbb{C}$ [9], for $\varepsilon > 0$ there exists a function $g \in C(F \cap \bar{D}_n)$ such that

$$\|f - g\|_{0, F \cap \bar{D}_n} < \varepsilon/2.$$

But, according to well-known extension theorems, g can be extend to $g^* \in C(F)$. So by hypothesis $g^* \in [M_F(\mathbb{C})]_{0, F}$ and by applying BMO-Runge's theorem there exists a rational function r without poles on $F \cap \bar{D}_n$ such that

$$\|g - r\|_{0, F \cap \bar{D}_n} < \varepsilon/2.$$

Therefore $f \in [R(F \cap \bar{D}_n)]_{0, F \cap D_n}$ and $f \in [M_F(\mathbb{C})]_{0, F}$ (Theorem 8). Now (9) follows from Theorem 10.

5. Entire Approximation in BMO-norms

With BMO-Runge's theorem we have seen that it can be advantageous to recolocate the poles of the approximating rational function without affecting the approximation itself. It is important for us that an analogous result holds for meromorphic functions. In order to prove a pole shifting theorem, we need the following lemma whose proof is similar to Lip α case, when considering the BMO-norm instead of Lip α-norm [4].

Lemma 12. *Let F be a closed subset of $\mathbb{C}$ and $f \in M_F(\mathbb{C})$. Suppose that f has a pole at $z_1 \in \mathbb{C}\backslash F$. If $\varepsilon > 0$ and $z_2 \in \mathbb{C}\backslash F$ with z_1 and z_2 being in the same connected component of $\mathbb{C}\backslash F$, then there exists a function $g \in M_F(\mathbb{C})$ such that:*

(a) *g has a pole at z_2.*

(b) *g is holomorphic at z_1.*

(c) *If z is a pole of g and $z \neq z_i$, for $i = 1, 2$, then z is also a pole of f.*

(d) $\|f - g\|_{0, F} < \varepsilon$.

Since, obviously, a relocation of infinitely many poles is required, it is essential that $\mathbb{C}^*\backslash F$ be locally connected, thus we obtain:

Theorem 13. *Let F be a closed subset in $\mathbb{C}$, $\mathbb{C}^*\backslash F$ being locally connected. Then for every $f \in M_F(\mathbb{C})$ and $\varepsilon > 0$ there exists a rational function r and an entire function h such that*

$$\|f - r - h\|_{0, F} < \varepsilon.$$

If $\mathbb{C}^*\backslash F$ is connected, we can choose $r \equiv 0$.

Proof. All the poles of f can be connected with $\{\infty\}$ by arcs contained in $\mathbb{C}\backslash F$ except at most a finite number of them. Moreover, according to [5, p. 138 ff.], we can choose the arcs γ_k such that if K is a compact subset of $\mathbb{C}$, then K intersects at most a finite number of these arcs.

Let $\{G_n; n \in N\}$ be an exhausting sequence of $\mathbb{C}$ by precompact domains with $\bar{G}_n \subset G_{n+1}$. Denote r the sum of the singular parts of f in the poles of f lying in the bounded components of $\mathbb{C}^*\backslash F$, and $g = f - r$. Since there only exists a finite number of γ_n intersecting $\bar{G}_1$ then we can displace the corresponding poles along these arcs out of $\bar{G}_1$ (Lemma 12). So there exists a $g_1 \in M_{F \cup \bar{G}_1}(\mathbb{C})$ such that

$$\|g_1 - g\|_{0, F} < \varepsilon/2$$

and all the poles of g_1 can be connected to $\{\infty\}$ by arcs $\gamma_k \subset \mathbb{C}\backslash F$ which do not intersect $\bar{G}_1$. By a recurrent procedure, it is shown that there exists a function $g_n \in M_{F \cup \bar{G}_n}(\mathbb{C})$ such that

$$\|g_n - g_{n-1}\|_{0,\, F \cup \bar{G}_{n-1}} < \varepsilon/2^n \tag{10}$$

and all the poles of g_n can be connected to $\{\infty\}$ by arcs which do not intersect $\bar{G}_n \cup F$.

We define the function $h = \lim g_n$. Thus h is entire in G, because

$$h = \lim_{n \to \infty} g_n = g_N + \sum_{n=N}^{\infty} (g_{n+1} - g_n)$$

and g_n is analytic in G_N for every $n \geqslant N$. By (10) the series converges in BMO-norm on $\bar{G}_N$ and $\sum_{n=N}^{\infty} (g_{n+1} - g_n)$ belongs to $A_0(\bar{G}_N)$ [7]; hence $h \in H(G_n)$ for each n. Moreover

$$\|f - r - h\|_{0,\, F} \leqslant \|g_1 - g\|_{0,\, F} + \sum_{n=2}^{\infty} \|g_n - g_{n-1}\|_{0,\, F} \leqslant \varepsilon.$$

If $\mathbb{C}^*\backslash F$ is connected, then $r \equiv 0$.

Thus we may extend Arakeljan's theorem (see [5, p. 142] for the uniform approximation) to BMO-norm, by obtaining sufficient conditions for the entire approximation

THEOREM 14. *Let F be a closed subset of $\mathbb{C}$. If $\mathbb{C}^*\backslash F$ is connected and locally connected and*

$$M_*^1(\Delta(z, \delta)\backslash F^0) \leqslant C M^1(\Delta(z, \delta)\backslash F)$$

for all z and all $\delta > 0$, then all $f \in A_0(F)$ can be approximated in BMO-*norm by entire functions.*

Proof. The proof is deduced form Theorem 10 and Theorem 13.

Note that the conditions of the above theorem are not necessary, since sets with measure 0 are ignored by BMO-norm. Thus we can construct a closed set F such that $F\backslash F_1$, F_1 being a measurable set with measure 0, has its complement $(\mathbb{C}^*\backslash(F\backslash F_1))$ connected and locally connected, although $\mathbb{C}^*\backslash F$ is not connected and locally connected.

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